Analytic Solutions of the Radiation Modes Problem and the Active Control of Sound Power

Cédric Maury$^{1,2}$ and Stephen J. Elliott$^1$

$^1$Institute of Sound and Vibration Research
University of Southampton, SO17 1BJ Southampton, United Kingdom
cm@isvr.soton.ac.uk, sje@isvr.soton.ac.uk

$^2$Laboratoire Roberval CNRS-UMR 6066, Secteur Acoustique
Université de Technologie de Compiègne, BP 20529, 60205 Compiègne, France
cedric.maury@utc.fr

Abstract

This study explores an application to vibro-acoustics of the so-called concentration problem: of determining which functions that are band-limited in one domain have maximal energy concentration within a region of the transform domain.

Analytic solutions to this problem are seen to involve prolate spheroidal wave functions. In particular, exact expressions are given for the radiation efficiencies and shapes of the radiation modes of a baffled beam. It is shown that a generalisation of the concentration problem to the two-dimensional case provides analytic solutions that solve with a good accuracy, although approximately, the radiation problem.

From the properties of these solutions, an upper limit has been found on the number of radiation modes to be controlled respectively for the beam case and for the panel case in order to significantly reduce the sound power radiated from these structures.

1. Introduction

The concentration problem arose about 40 years ago in Fourier analysis applied to the theory of communication in electrical engineering. The problem was to minimize signal "spillovers" both in time and frequency domains. It can be shown that it is equivalent to find a class of band-limited signals that would possess the largest fraction of energy in a given time interval. This question has been elegantly answered by Slepian and his collaborators in a remarkable series of papers [1,2,3]. The answer is based on the fact that these signals are the eigenfunctions of a convolution integral operator that commutes with a much simpler second order differential operator. For this reason, the former operator has the same eigenfunctions as the latter: they consist of a set of special functions, the prolate spheroidal wave functions, which therefore solve the concentration problem.

In this study, we underline the close relationship that exists between the concentration problem and an important problem arising in acoustics, namely the formulation of the total acoustic power radiated by a planar structure in terms of its radiation modes [4,5]. This problem addresses the possibility of finding a set of independent optimal velocity distributions on the surface of a planar structure, also known as radiation modes, that best represent the sound power radiated among all possible velocity patterns. When suitably formulated, this problem involves the maximisation of a cost function with the same integral operator as the one that occurs in the concentration problem [1]. From this formal similarity, we are then able to formulate exact solutions for the acoustic radiation problem. This solution can be used to provide a closed form solution for the radiation modes and radiation efficiencies of a baffled beam.

2. The concentration problem

One natural question that has been considered by Slepian et al. in the context of communication theory is to determine which band-limited signal, $x_k(t)$ whose Fourier transform, $X(f)$, is zero-valued for $|f| > K$, has maximal energy concentration in a given time slot $(-T/2, T/2)$, i.e. for which band-limited functions the ratio

$$\rho(T) = \frac{\int_{-T/2}^{T/2} x_k^2(t) dt}{\int_{-\infty}^{\infty} x_k^2(t) dt}$$

is maximised. Making use of the Parseval theorem, the ratio (1) written in the frequency domain, is maximised when $X(f)$ satisfies, for all $|f| \leq K$, the following homogeneous Fredholm integral equation:
\[
\int_{-\infty}^\infty \frac{\sin \pi T (f - f')}{\pi (f - f')} X(f')df' = \rho(T)X(f)
\] (2)

The solution to equation (2) is given in section 3. The notations used for the signal processing formulation are summarized in Figure 1(a) in relation with the notations used for vibro-acoustic problem (Figure 1(b)).

![Figure 1: Notations for the concentration problem.](image)

3. Prolate spheroidal wave functions

After normalisation, equation 2 can be written as:

\[
\int_{-\infty}^1 \frac{\sin \alpha(s-s')}{\pi(s-s')} \psi(s')ds' = \hat{\lambda}(\alpha) \psi(s), \quad |s| \leq 1,
\] (3)

where \( \alpha = \pi KT \). It has been shown by Slepian [1] that the angular prolate spheroidal wave functions \( S_{\alpha n}(\alpha,s) \) are the eigenfunctions of the finite convolution integral operator (3) such that, for all \( s, \) real or complex, and for all integer \( n: \)

\[
\int_{-\infty}^1 \frac{\sin \alpha(s-s')}{\pi(s-s')} S_{\alpha n}(\alpha,s')ds' = \frac{2\alpha}{\pi} R_{\alpha n}(\alpha,1) S_{\alpha n}(\alpha,s),
\] (4)

where \( R_{\alpha n}(\alpha,s) \) is known as a radial prolate spheroidal wave function and differs from \( S_{\alpha n}(\alpha,s) \) by a scale factor. The eigenvalues, \( \lambda_n(\alpha) = 2\alpha R_{\alpha n}(\alpha,1)^2/\pi \), can be seen as the portion of the energy of \( \psi_n \equiv S_{\alpha n} \) that is concentrated within the interval \((-1,1)\). After ordering the eigenvalues, say \( \lambda_0 \geq \lambda_1 \geq \cdots \), it is the band-limited eigenfunction \( \psi_n \) associated to the largest eigenvalue that has most energy in \((-1,1)\) of all band-limited functions, and so on.

As shown in Figure 2, the concentration of energy of the first eigenfunctions within \((-1,1)\) increases with \( \alpha \). For any given \( \alpha \), the number of eigenfunctions whose energy lies in \((-1,1)\), i.e. the “most energetic” eigenfunctions, is \( 2\alpha/\pi \), thus providing a criterion to determine the number of band-limited signals which has the largest energy concentration within a given time slot.

![Figure 2: The first eigenvalues \( \lambda_n(\alpha) \) for \( \alpha = 8 \) (dashed curve) and for \( \alpha = 16 \) (solid curve).](image)

4. Radiation modes of a baffled beam

Let consider the acoustic pressure radiated by a harmonically-excited elastic beam of length \( L \) and width \( w \), set in an infinite baffle and in contact with a fluid at rest with density \( \rho \) and sound speed \( c \). The in-phase acoustic wall pressure field radiated by each radiation mode, \( \psi_n \), has the same spatial distribution as its corresponding surface velocity. The following integral equation is satisfied:

\[
\frac{\rho c k_n w}{2} \int_0^1 \frac{\sin k_n(x-x')}{\pi(x-x')} \psi_n(k_n,x')dx' = \lambda_n(k_n)\psi_n(k_n,x),
\] (5)

where the left hand side represents the real part of the acoustic pressure field radiated by the \( n^{th} \) radiation mode, \( \lambda_n \) is the corresponding eigenvalue and \( k_n \) is the acoustic wavenumber. It can be seen that the integral equation (5) has the same kernel as equation (3) and so, the radiation modes of a baffled beam and their corresponding radiation efficiencies, \( \sigma_n \), have the following analytical expression:

\[
\psi_n(k_n,x) = S_{\alpha n}\left(\frac{k_nL}{2},\frac{2x}{L}-1\right),
\] (6)

\[
\sigma_n(k_n) = \frac{w(k_nL)}{4\pi} R_{\alpha n}\left(\frac{k_nL}{2},1\right).
\] (7)

where \( \sigma_n(k_n) = 2\pi c\lambda_n(k_n)/\rho cL \). The radiation modes associated to the largest eigenvalues correspond, in relation with Figure 1(b) and Figure 2, to the velocity distributions specified over the length of the beam and
which have maximal energy concentration in a given radiation bandwidth \((-k_w, k_w)\). Only these “supersonic” acoustic radiation modes with high radiation efficiencies contribute significantly to the sound pressure radiated in the far-field.

To the best of the authors’ knowledge, no exact expressions has so far been reported over the whole frequency range for the radiation efficiencies and radiation modes of a baffled beam.

Figure 3: The radiation efficiencies of the first three radiation modes of a baffled beam (bold), their numerical approximations (thin) and their low-frequency (dashed) and high-frequency (dash-dotted) asymptotics.

Figure 3 presents the radiation efficiencies of the first three radiation modes determined through the analytic expression (7) together with their low- and high frequency asymptotics for the ratio \(w/L = 1/64\), as a function of the dimensionless frequency \(k_w L\). Their values coincide with those obtained numerically from an eigenvalue decomposition of the radiation matrix for an array of 64 discrete elements regularly spaced along the length of the beam [5].

Figure 4 depicts the shape of the first four radiation modes determined through equation (6) and for an excitation frequency corresponding to \(k_w L = 10\). These ‘exact’ eigenfunctions coincide with those numerically calculated by Elliott & Johnson [5].

5. Radiation modes of a baffled panel

A generalization of the present work to the two-dimensional case is now presented. Assuming separation of variables, the radiation modes and the radiation efficiencies of a baffled panel of length \(L\) and width \(w\) can be approximated by:

\[
\psi_{mn}(k_w, x, y) \approx \sin \left( \frac{k_w L}{2} \right) \sin \left( \frac{2x}{L} \right) \sin \left( \frac{2y}{w} \right),
\]

\[
\sigma_{mn} (k_w) \approx \frac{(k_w L)(k_w w)}{\pi^2} R_{mn} \left( \frac{k_w L}{2} \right)^2 R_{mn} \left( \frac{k_w w}{2} \right)^2.
\]

Figure 4: The mode shapes of the first four radiation modes of a baffled beam for an excitation frequency corresponding to \(k_w L = 10\).

Figure 5 shows the radiation efficiencies of a panel, with \(w/L = 0.57\), that have been calculated numerically for the true kernel and analytically for its approximate form (9), as a function of \(k_w L\). It can be seen that, for acoustic wavelengths longer than twice the panel dimensions, say \(k_w L \leq \pi\), the analytic approximation (9) represents with a reasonable accuracy the first acoustic radiation efficiencies of the panel. However, discrepancies due to the assumption of variable separation occur at higher frequencies for the first radiation modes and at moderate frequencies for higher order radiation modes.

Figure 5: The radiation efficiencies of the first six radiation modes of a baffled panel (bold) and their analytical approximations (dashed).

6. Physical limitations for the active control of sound power

Assuming that each of the radiation modes is excited to an equal amplitude, an upper bound can be determined on the number of radiation modes that independently contribute to the sound power radiated from planar rectangular structures at a given frequency.
This could provide some guidelines on the maximum number of independent channels that could be used to drive a set of distributed actuators for actively canceling the contribution of the first radiation modes to the sound radiated by vibrating structures under general excitation conditions.

![Graph](image)

**Figure 6**: The number of radiation modes required to account for 90% of the sound power radiated from a 1 m length beam (crosses) and from a rectangular panel \((w/L = 0.57)\) when using either the full form (circles) or the separated form (stars) of the radiation operator.

For the beam case, for large \(k_w L\), most of the radiated power is captured by the first \(k_w L/\pi\) radiation modes in agreement with the results shown in Figure 2. For the panel case, if we assume that the radiation operator can be written in a separable form, the latter criterion can be generalised and it is found that at most \((k_w w)(k_w L)/\pi^2\) radiation modes are required to account for about 90% of the radiated power. As noted earlier, the approximate form (8) is reasonably accurate at low frequencies and so, the analytic criterion derived for the panel case is only relevant in the low frequency domain, say up to about \(k_w L \approx 8\). However, at higher frequencies, it overestimates the required number of radiation modes that has been calculated numerically in the full case. This is not surprising since, at high frequencies, the true radiation operator is more correlated over the panel surface than the separated one and so, a smaller number of independent radiation modes is sufficient for its representation.

### 7. Conclusions

Analytic approximations have been presented for both the radiation modes and the radiation efficiencies of either a baffled beam or a baffled panel in terms of prolate spheroidal wave functions. Comparisons with the numerical solution have shown that the approximate solution is valid up to about \(k_w L = \pi\). In order to reduce the sound power radiated from the structure by 10dB, it is found that an upper limit on the number of radiation modes to be controlled is about \(k_w L/\pi\) and \((k_w w)(k_w L)/\pi^2\), respectively for the beam case and for the panel case. The closed-form solutions that we have derived for the radiation mode shapes could be incorporated into the design of radiation filters in order to provide, from velocity measurements, an accurate estimation of the sound power radiated from the structure in the low-frequency domain. In the control system, this estimate could then be used as an input to a feedback compensator to generate the signals driving a set of actuators.

Finally, it can be seen that the radiation operator in the expression for the structural acoustic power radiated is formally similar to the kernel associated to the averaged mean-square incident pressure due to an acoustic diffuse field impinging on a wall [6]. Then, the analytic criteria we have found could also be used to determine, in a different context, a complete set of independent modes that best reproduce a given diffuse wall-pressure field over the surface of a structure. This idea would be relevant to the laboratory simulation of acoustic diffuse wall-pressure fields, in order to compensate at low frequencies for the poor reverberant characteristics of the testing room [7].

### 8. References


