On Study of the Theory and Algorithm of the Three Dimensional Coupled Mode – Parabolic Equation

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Abstract

The theory and algorithm of the three dimensional coupled mode-parabolic equation (CMPE3D) are studied in this paper. The CMPE3D solution of the sound field in underwater acoustic waveguides due to a time-harmonic source is a hybrid model expressed in terms of the normal modes in vertical direction and the mode amplitude coefficients in horizontal directions. The mode amplitude coefficients are solved by using a PE approach. By using the generalized phase integral (WKBZ) theory and the Beam-Displacement Ray-Mode (BDRM) theory, the normal mode analysis can be processed efficiently. A PE-based algorithm, which was implemented in model FOR3D by Lee D. et al is adopted to solve the mode amplitude coefficients. Taking advantage of the merits of the couple mode and the PE approximations, the CMPE3D can predict the sound field efficiently with high precision. Numerical simulations indicate that for some problems, the efficiency has been greatly improved by using CMPE3D instead of the PE approximations.

1. Introduction

Modeling sound propagation in three dimensional environments has received considerable attention in the ocean acoustic community. Some models based on the coupled normal modes or PE methods have been developed. Abawi et al[1] have developed a coupled mode parabolic equation (CMPE3D) model, which is a generalization of the adiabatic mode PE that includes mode coupling terms. The CMPE3D is usually more efficient than three-dimensional PE models because the number of propagating modes is usually much smaller than the number of grid points that are required in a finite-difference treatment of the depth operator.

A fast algorithm of CMPE3D is presented in this paper. The CMPE3D solution of the sound field due to a time-harmonic point source with time dependence factor of \( \exp(-i\omega t) \) can be expressed in terms of normal modes and mode amplitude coefficients.

\[
p(r, \theta, z) = \rho^{1/2} \sum_{n=1}^{\infty} (k_n r)^{-1/2} \phi_n(r) \Theta_n(\theta) \Phi_n(z),
\]

where the three dimensional pressure \( p \) is function of range \( r \), azimuth \( \theta \), and depth \( z \). Density \( \rho(z) \) is function of depth. \( k_n \) is the horizontal wave number and \( \beta_n \) is the mode attenuation. Local mode analysis will given the values of \( k_n \) and \( \beta_n \). The improved WKBZ theory[2, 3] and BDRM[4] theory is used to efficiently analyze the local modes.

The \( n \)th mode amplitude coefficient \( \phi_n(\theta, r) \) in Eq. (1) satisfies the coupled mode coefficient equations. Abawi et al[1] used PE approach to numerically solve the coupled mode coefficient equations. As a result, the \( \phi_n(\theta, r) \) can be solved from following matrix equations

\[
\frac{\partial \Phi}{\partial r} = D\Phi - i\mu\Phi + i\omega^2 \Phi + k^2 \Phi, \quad (2)
\]

where the elements of vector \( \Phi \) is \( \phi_n \), matrix \( k \) and \( \mu \) are diagonal matrices with diagonal elements of \( k_n \) and \( \mu_n \), respectively. The elements of matrix \( D \) are defined as
\[ d_{mn} = (N_{mn} + S_{mn}) \exp(i \int_{0}^{L}(\mu_n - \mu_m)dr) \left( k_m^2 - k_n^2 \right). \]  
(3)

where
\[ N_{mn} = \frac{\partial H}{\partial r} \left[ \phi_{m2/1} \left( \gamma_m^2 \frac{\partial^2 \phi_{m2/1}}{\partial \theta^2} + \gamma_m^2 \frac{\partial^2 \phi_{m2/1}}{\partial z^2} \right) + \frac{\partial \phi_{m2/1}}{\partial z} \left( 1 - \frac{\rho_2}{\rho_1} \right) \right], \]
(4)

and
\[ S_{mn} = \int_{0}^{\infty} (\partial k^2/\partial r) \phi_{m2/1} \phi_{n2/1} dz. \]  
(5)

The subscripts 1 and 2 of Eq. (4) denote the parameters above and below the water-seabed interface, respectively. \( \gamma_1 = \sqrt{k_1^2 - k_{n1}^2} \) and \( \gamma_2 = \sqrt{k_2^2 - k_{n2}^2} \) are the vertical wavenumbers above and below the water-seabed interface. Note that the half-space liquid seabed is assumed in this paper. For multilayer sediment seabed model, the coupling term of Eq. (4) should be changed. Equation (2) can be solved by using split step method, which involves solving two equations as following:
\[ \frac{\partial \Phi}{\partial r} = D \Phi, \]  
(6)

and
\[ \frac{\partial \Phi}{\partial r} = -i \mu \Phi + \left( i + \Delta \Phi^2 \right) \frac{\partial^2 \phi}{\partial \theta^2} \Phi. \]  
(7)

Using the Crank-Nicolson integration, Eq. (6) can be solved using implicit finite difference method:
\[ \Phi^{n+1} = \left( 1 - \Delta \Phi^2 / 2 \right) \Phi^{n} + \left( i + \Delta \Phi^2 / 2 \right) \Phi^{n}. \]  
(8)

The matrices of Eq. (7) are all diagonal matrices, so Eq. (7) can be rewritten as:
\[ \frac{\partial \phi_m}{\partial r} = i \mu_m \left( 1 + \sqrt{1 + X} \right) \phi_m, \ m = 1, 2, \cdots, N, \]  
(9)

where \( X = (r^2 \mu_m)^{-1} \delta^2 / \partial \theta^2 + \eta_m \) and \( \eta_m = (k_m^2 - \mu_m^2) / \mu_m^2. \)

Equation (9) can be solved using a wide angle PE approach based on the implicit finite difference method, which was developed by Ding Lee[5],
\[ (1 + \delta X) \phi_m^{r+1} = (1 + \delta X) \phi_m^r, \]  
(10)

where \( p = (1 + \delta) / 4, \ \bar{p} = (1 - \delta) / 4 \) and \( \delta = i \mu_m \Delta r. \)

The geometry of the finite difference discretizations for a cylindrical wedge is shown in Fig. 1. The wedge is discretized to \( L \) parts in the azimuthal direction. The index is clockwise increasing with increment of \( \Delta \theta. \) The first part and the last part involve the side wall boundary conditions. For waveguide with 360 degrees in the azimuthal direction, the periodicity condition is used instead of the side wall boundary conditions, i.e. \( \phi_n(r, \theta + 2\pi) = \phi_n(r, \theta). \) The mode amplitude coefficient at range \( r + \Delta r \) is predicted when the field at range \( r \) has been solved, and then the field at \( r + 2\Delta r \) is predicted. In this manner, the model marches forward one step at a time, until maximum range has been reached.

### 3. Example

Numerical simulation of the sound propagation in a wedge waveguide with a circle upheaval is illustrated as an example. The schematic geometry of the waveguide is shown in Fig. 2 and Fig. 3. The initial water depth is 60m increasing linearly to 100 m at a range of 10 km. A point source is placed at 50m and the receiver depth is 30m. The parameters used for simulation are list in Table 1. The sound velocity of homogenous water volume is 1480 m/s. Pressure-release surface and half space liquid seabed are assumed. The seabed sound velocity is 1587 m/s. The density ratio between the seabed and the water is 1.8 and the seabed attenuation is 0.3 dB/wavelength.

<table>
<thead>
<tr>
<th>Table 1 Parameters used for numerical simulation</th>
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<tr>
<td>Frequency: 30, 60, and 100 (Hz)</td>
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<tr>
<td>Source depth: 50 m</td>
</tr>
<tr>
<td>Receiver depth: 30 m</td>
</tr>
<tr>
<td>Water sound speed: 1480 m/s</td>
</tr>
<tr>
<td>Seabed sound speed: 1587 m/s</td>
</tr>
<tr>
<td>Density ratio: 1.8</td>
</tr>
<tr>
<td>Seabed attenuation: 0.3 dB/wavelength</td>
</tr>
</tbody>
</table>

The TLs along azimuths of 0 ~ 180 degree at frequencies of 30, 60 and 100Hz are predicted, and the TLs at frequencies of 30, 60 and 100 Hz are plotted in Fig. 4 and Fig. 5. In Fig. 4, it is shown the TLs as function of range at azimuth of 0, 30, 60, and 90 degree. Since the TL curves are symmetric, only TLs at azimuth
between 0 ~ 90 degree are shown. The TLs as function of the azimuth at ranges of 3 km and 8 km are shown in Fig.5. The solid lines in Fig. 4 and Fig. 5 are predicted by CMPE3D, and the dashed lines are predicted by FOR3D. The computational time is listed in Table 2.

### Table 2: Comparison of computational time

<table>
<thead>
<tr>
<th>f (Hz)</th>
<th>CMPE3D (Second)</th>
<th>FOR3D (Second)</th>
<th>Time ratio</th>
</tr>
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<tr>
<td>100</td>
<td>51.30</td>
<td>20539.11</td>
<td>1/400</td>
</tr>
<tr>
<td>60</td>
<td>31.42</td>
<td>2921.05</td>
<td>1/93</td>
</tr>
<tr>
<td>30</td>
<td>11.64</td>
<td>1493.15</td>
<td>1/128</td>
</tr>
</tbody>
</table>

From Fig. 4 and Fig. 5, one can see that the TLs predicted by CMPE3D are in good agreement with those predicted by FOR3D, except for TLs in Fig.5(c) at some azimuths. The reason of the large error (about 2 dB) in Fig.5(c) is due to the large grid step for FOR3D used in the depth direction. One of the shortcomings of the three dimensional PE approach is that the grid points in the depth direction increase greatly with the frequency increasing. Therefore, much computer memory is needed for higher frequency. There is no enough memory to support grid points in the depth direction for FOR3D to precisely predict the TLs at frequency of 100 Hz. Since number of local modes is much less than that of the grid points in the depth direction used by FOR3D, the CMPE3D has no such memory problem.

The computational time list in Table 1 shows that the computational speed of CMPE3D is much faster than that of FOR3D. The time ratio is 93 for frequency of 60 Hz, and 400 for frequency of 100 Hz. The computational time of FOR3D for frequency of 100 Hz is 7 times of that for frequency of 60 Hz. The reason is that too much memory being used will slow the computational speed.
4. Conclusions

The theory and algorithm of the three dimensional coupled mode parabolic equation (CMPE3D) are studied in this paper. On the basis of the generalized phase integral (WKBZ) theory, and beam-displacement ray mode (BDRM) theory, a fast algorithm of the CMPE3D is presented. A wide angle PE algorithm is also used to solve the mode amplitude coefficient equations.

Numerical simulation indicates that the TLs predicted by CMPE3D are in good agreement with those predicted by FOR3D, and the computational speed of CMPE3D is much faster than the computational speed of FOR3D.

One of the shortcomings of the PE method is that for higher frequency, more grid points and more memory are needed, which leads to the longer computational time. Much less memory being needed by CMPE3D gives its ability to treat higher frequency propagation problem than three dimensional PE model.

5. Acknowledgements

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6. References


