Imaging localized hysteretic damage using Nonlinear Resonance Ultrasound Spectroscopy (NRUS)

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Abstract
A bar containing a defect is modeled by assuming linear acoustical properties everywhere except in the damaged region where the stress-strain relation is hysteretic and nonlinear. Generalized and analytical NRUS equations are developed which predict the shift in resonance frequency, and the creation of harmonics as a result of the nonlinear defect. A good agreement with numerical simulations is found.

1. Introduction
Linear RUS is a well-known technique which is applied to invert the elasticity tensor of anisotropic media from the acoustical resonance spectrum [1,2]. Recently a nonlinear version was presented [3]. In this paper, we extend the NRUS approach to the case of nonlinear hysteresis. The analytical predictions are independently tested by EFIT-simulations [4].

2. Theory
We first derive the general equations for one dimensional (1D) resonant bar modeling, which are valid for all modes. Then, we apply these results to find the amplitude dependence of the resonance frequency and of the response amplitudes of particular modes for a given external forcing.

2.1. General 1D NRUS equations for hysteretic nonlinearity
In accordance with the concept of a bow-tie type modulus-strain relation [5], the macroscopic 1D stress-strain relation in the case of hysteretic nonlinearity can be expressed by

$$\sigma = K \varepsilon - K \alpha \Delta \varepsilon \cdot \varepsilon - \frac{1}{2} \text{sgn} \left( \Delta \varepsilon \right) \left( (\Delta \varepsilon)^2 - \varepsilon^2 \right),$$  \hspace{1cm} (1)

where $\sigma$ is the stress, $\varepsilon$ the strain, $\Delta \varepsilon$ the amplitude of the strain, $K$ the linear stiffness and $\alpha$ a constant describing the strength of the hysteresis. We will suppose in the following that the damaged region in a bar can be modeled by giving it a nonlinear hysteretic behavior. In the absence of damping and external forcing, a 1D bar consisting of a material cells for which the hysteretic stress-strain relation (1) applies, will support waves governed by the following equation:

$$\rho \frac{\partial^2}{\partial x^2} u = \partial_t \sigma$$ \hspace{1cm} (2)

with $\rho$ the density, $u$ the displacement field which is connected to the strain $\varepsilon$ by

$$\varepsilon = \partial_t \sigma \cdot u,$$  \hspace{1cm} (3)

and $x$ is the spatial coordinate.

If we combine Eqs. (1), (2) and (3), we obtain a nonlinear partial differential equation for $u$. In order to solve this equation, we decompose $u$ into

$$u = \sum_{m} \varphi_m(x) z_n(t),$$  \hspace{1cm} (4)

where

$$\varphi_m(x) = \cos \left( \frac{m \pi x}{L} \right)$$  \hspace{1cm} (5)

is the displacement field of the $m^{th}$ mode of the bar in the linear case ($\alpha=0$). (We assume that the bar has length $L$). Substituting Eq. (4) into Eq. (3), multiplying it by $\varphi_n(x)$ and then integrating over $x$, one obtains a set of nonlinear coupled ordinary differential equations in the unknowns $z_n$:

$$\partial_t^2 z_n + \omega_n^2 z_n = B_{np} \left[ \text{sgn} \left( \partial_t \varphi_p \right) \left( (\Delta \varepsilon_p)^2 - \varepsilon_p^2 \right) \right. - 2 \Delta \varepsilon_p \varepsilon_p \right]$$ \hspace{1cm} (6)

where $\Delta \varepsilon_p$ is the amplitude of mode $p$,

$$\omega_n^2 = \left( \frac{n \pi}{L} \right)^2 \frac{K}{\rho}$$ \hspace{1cm} (7)

is the linear circular resonance frequency and

$$B_{np} = - \frac{K}{\rho L} \int_0^L dx \alpha \partial_t \varphi_p \left[ \partial_t \varphi_p \right]$$ \hspace{1cm} (8)

is a nonlinear coupling coefficient describing the coupling of mode $p$ to mode $n$. In deriving Eq. (6), we made the assumption that only mode $p$ is excited at low amplitude levels. The equation clearly shows that due to the nonlinear interaction mode $n$ can be generated by mode $p$ which acts as a source. Eq.(6) can be modified to include damping and external forcing. This yields the following equations:

$$\partial_t^2 z_n + 2 \frac{\partial}{\partial \Omega} \partial_t z_n + \left[ (\Omega)^2 - 2 n \Omega \Delta \varepsilon_n \right] z_n = F \cos(\Omega t) +$$

$$B_{np} \left[ \text{sgn} \left( \partial_t \varphi_p \right) \left( (\Delta \varepsilon_p)^2 - \varepsilon_p^2 \right) \right. - 2 \Delta \varepsilon_p \varepsilon_p \right]$$ \hspace{1cm} (9)
Here, $Q$ is the quality factor (inverse attenuation), $\Omega$ the driving frequency, $F$ the driving amplitude and $\delta_n = n\Omega - \omega_n$ describes the mismatch between $n$ times the excitation frequency and the $n^{th}$ mode. The term in $\delta_n^2$ has been neglected, since we assume that the excitation frequency is close to the linear resonance frequency of mode $p$. (This is consistent with the above mentioned assumption that only mode $p$ provides a contribution to the nonlinear term in Eq. (6)). Equation (9) can be solved by techniques from nonlinear dynamics [1]. In general, one decomposes the time functions $z_p(t)$ into

$$z_p(t) = \frac{1}{2} \left( a_p e^{i\omega t} + \bar{a}_p e^{-i\omega t} \right),$$

$$z_{p\omega} = \frac{1}{2} \left( a_{p\omega} e^{i\omega t} + \bar{a}_{p\omega} e^{-i\omega t} \right),$$

where the bar-symbol indicates complex conjugation. Restricting the analysis to slow variations in the amplitudes $a_p$ and after some algebraic manipulations (including the decomposition of the $sgn$ function in its Fourier components) we have found the following equations for the mode amplitudes when $n = 1, 2$ and $3$:

$$\frac{da_p}{dt} = -i\delta_p a_p - \frac{\Omega}{Q} a_p - iF - \frac{B_{pp}}{2\Omega} a_p^2 \left( \frac{4}{3\pi} + i \right) |a_p| a_p,$$  

$$\frac{da_{p\omega}}{dt} = -i\delta_{p\omega} a_{p\omega} - \frac{2\Omega}{Q} a_{p\omega},$$

$$\frac{da_{3p}}{dt} = -i\delta_{3p} a_{3p} - \frac{3\Omega}{Q} a_{3p} - \frac{4}{45\pi \Omega} B_{pp} a_p^3.$$  

Equations (11), (12) and (13) govern the response behavior of mode $p$, its second and third harmonic for an external excitation with amplitude $F$ and circular frequency $\Omega$.

### 2.2. Response of mode $p$

The complex valued amplitude response of the bar at the frequency of the fundamental mode $p$ is given by Eq. (11). It can be solved by decomposing

$$a_p = A_p e^{\phi_p},$$

which leads to

$$\frac{dA_p}{dt} = -\frac{\Omega}{Q} A_p - F - \frac{B_{pp}}{2\Omega} A_p^3,$$  

$$\frac{d\phi_p}{dt} = -\delta_p - \frac{F}{2\Omega} \cos \phi_p + \frac{B_{pp}}{\Omega} A_p.$$  

In a classical SimonRus experiment [6] the steady state response of the bar is measured. Under these conditions $A_p$ and $\phi_p$ remain constant and hence the right hand side of Eqs. (15) and (16) should be zero. This yields two algebraic equations in the unknowns $A_p$ and $\phi_p$. After elimination of the phase, we obtain

$$\left( \frac{4B_{pp} A_p - \Omega}{3\pi Q} \right)^2 + \left( \frac{B_{pp} A_p - \delta}{2\Omega} \right)^2 = \left( \frac{F}{2\Omega} \right)^2.$$  

Equation (17) is of practical interest since it describes the shape of the resonance curves: the response amplitude $A_p$ as function of the driving frequency $\Omega$ at the forcing amplitude $F$. For a fixed $F$, the maximum of the resonance curve determines the resonance frequency at that driving level. Thus by looking for the drive frequency $\Omega$ which maximizes $A_p$, we obtain the excitation amplitude dependence of the resonance frequency. To first order in $B_{pp}$ one finds

$$\frac{\Delta \Omega}{\omega_p} = \left( 1 + \frac{8}{3\pi Q} \omega_p^2 \right) A_p,$$  

where $\Delta \omega_p$ is the shift in resonance frequency with respect to the linear (or low amplitude) resonance frequency. If we assume that the nonlinearity $\alpha$ is constant and only nonzero in a small region $[x_0+d/2, x_0+d/2]$, with $d$ much smaller than the modal wavelength, then $B_{pp}$ takes on the form:

$$B_{pp} = -\frac{K}{\rho L} \delta \left( \frac{p\pi}{L} x \right) \sin \left( \frac{p\pi}{L} x \right).$$  

Considering a localized nonlinearity in the bar, we substitute expression (19) into Eq. (18) to find the shift in the resonance frequency (we neglect the term in $Q$ of Eq. (18) as it is much smaller than 1 for moderate $Q$-values):

$$\frac{\Delta \omega}{\omega_p} = -\alpha \frac{d}{L} \sin \left( \frac{p\pi}{L} x \right) \left| e_p \right|,$$  

where

$$e_p = \frac{p\pi}{L} A_p,$$  

is the strain response amplitude of the mode $p$. Eq. (20) shows first of all that the shift in resonance frequency is always negative, i.e. a softening occurs, which is usual for hysteretic nonlinearities. In addition, the shift is proportional to the strength of the nonlinearity $\alpha$, the effective width of the damage $d/L$, and linear with the strain amplitude (typical for a hysteretic nonlinearity). Finally the shift has a specific sensitivity to the location of the defect. The sensitivity factor behaves as the third power in the normalized strain field at the position of the damage, which originates from the nonlinear coupling coefficient $B$ expressed in Eq. (8). The largest coupling coefficient correspond to locations of maximal (normalized) strains fields. In general, it is obvious that only locations of nonzero normalized strains $\hat{e}_p \phi_p$ at the position of the defect (where $\alpha \neq 0$) will produce nonzero coupling coefficients, and as a consequence nonzero resonance frequency shifts.
2.3. Response of second harmonic

The response of the bar at the second harmonic (i.e. two times the frequency of the fundamental mode \( p \)) is given by Eq. (12). Since there is neither a linear forcing term nor a nonlinear source term in Eq. (12) its solution is

\[
a_{p}^{2} = 0.\tag{22}
\]

Indeed, the forcing which occurs at the fundamental frequency \( \Omega \) does not produce a source term for the second harmonic, which will therefore not be excited. This behaviour is consistent with the hysteretic nature of the nonlinearity [5]: hysteresis only generates odd harmonics in the response spectrum. It can be easily shown from Eq. (9) that all even harmonics of \( \Omega \) have zero amplitude: i.e. the even harmonics don’t get generated.

2.4. Response of third harmonic

The response of the bar at the third harmonic (i.e. 3\( \Omega \)) is given by Eq. (13). This equation can be solved in the same manner as for the fundamental, i.e., by decomposing the complex amplitudes in its modulus and its phase

\[
\begin{align*}
a_{p} & = A_{p} e^{i \phi_{p}}. \tag{23} \\
\alpha_{3p} & = A_{3p} e^{i \phi_{3p}}. \tag{24}
\end{align*}
\]

Substituting Eq. (24) and (23) into Eq. (13) leads to

\[
\begin{align*}
\frac{dA_{3p}}{dt} & = \frac{3 \Omega}{Q} A_{3p} - \frac{4}{45 \pi \Omega} \cos(3 \phi_{p} - \phi_{3p}) B_{3p} A_{p}^{2} \tag{25} \\
\frac{d\phi_{3p}}{dt} & = -\delta_{3p} - \frac{4}{45 \pi \Omega} \sin(3 \phi_{p} - \phi_{3p}) B_{3p} A_{p}^{2}. \tag{26}
\end{align*}
\]

Following the same reasoning as above, we obtain two algebraic equations in two unknowns \( A_{3p} \) and \( \phi_{3p} \). The phase can again be eliminated, and the result is

\[
\left( \frac{4 B_{3p} A_{p}^{2}}{45 \pi \Omega A_{3p}} \right)^{2} = \left( \frac{3 \Omega}{Q} \right)^{2} + (3 \Omega - \omega_{3})^{2}. \tag{27}
\]

Neglecting the second term in the right hand side (\( Q=80 \), and \( 3 \Omega - \omega_{3} \approx 10 \)) one arrives at the following expression for the third harmonic in terms of the strains and for the same constant and localized nonlinearity as in the previous case:

\[
\varepsilon_{3p} = \frac{4}{15 \pi L d Q} \sin \left( 3 p \frac{\pi}{L} x \right) \sin^{3} \left( p \frac{\pi}{L} x \right) \varepsilon_{p}^{2}. \tag{28}
\]

Interpreting Eq. (28), we first observe that the third harmonic amplitude is proportional to the strength of the nonlinearity \( \alpha \), the effective width of the damage \( d/L \) and to the inverse attenuation (or quality factor) \( Q \). Secondly the harmonic is quadratic in the strain amplitude (typical for a hysteretic nonlinearity). Finally the third harmonic amplitude has a sensitivity factor consisting of the product of the strain field of the third mode with the square of the strain field of the fundamental, both evaluated at the position of the damage. This dependency originates from the nonlinear coupling coefficient \( B_{3p} \) as seen in Eq. (8).

2.5. Response of fifth harmonic

In analogy with the above interpretations of formulas (20) and (28), one can expect the amplitude of the fifth harmonic to be of the form

\[
\varepsilon_{5p} = C_{5} e^{2} \tag{29}
\]

where \( C_{5} \) is a constant.

3. Discussion

We will now compare results from a numerical modeling using a multiscale finite difference approach [4] with our analytical calculations. We consider a bar with a single damaged zone located at a varying position in the bar. The values used in the simulations are: \( L=0.25 \text{ m}, \ K=10 \text{ GPa}, \rho=2600 \text{ kg/m}^{3}, \ d=L/20, \alpha=2000, \text{ and } Q=80.\)

Our analytical calculations (Eq. (20)) showed that the resonance frequency shifts by

\[
\frac{\Delta \omega}{\omega} = -\alpha \frac{d}{L} \sin \left( \frac{p \pi}{L} x \right) \varepsilon_{p} = -C_{1} \varepsilon_{p}. \tag{30}
\]

where \( C_{1} \) is called the shift proportionality coefficient. As noted before, \( C_{1} \) is clearly a nonlinear function of the position of the damage in the bar. The analytical expression of the coefficient can be compared with the analysis of the frequency shift obtained from numerical simulations. The analytical and numerical behavior are both plotted in Fig. 1 for the fundamental mode \( p=1 \).

The numerical predictions and the analytical formula of Eq. (30) agree very well. It is seen that the modal shift is more sensitive when the defect is located near the center of the bar. This is perfectly understandable since the mode with \( p=1 \) has a strain field proportional to \( \sin(\pi x/L) \) which means that the strain is maximal at the center. Therefore defects at the center will be felt optimally whereas defects close to the edges will not be triggered as there is no modal strain present there to activate the nonlinearity.

The analytical result expressed in Eq. (28) suggested that the amplitude of the third harmonic is of the form

\[
\varepsilon_{3p} = C_{3} \varepsilon_{p}^{2}. \tag{31}
\]

where \( C_{3} \) is the third harmonic proportionality factor. Again we compared the position dependent behavior of this coefficient with the results from a numerical simulation. These results are plotted in Fig. 2 for the fundamental mode \( p=1 \). A good agreement is observed. It is clear that the third harmonic amplitude is more sensitive to defects located at a position where both the strain fields of mode \( p=3 \) and \( p=1 \) are maximal. When the damage is located at nodes of the third mode, it is
not sensed by $C_3$. Thus, the defect can be seen as the region where the third mode interacts with the fundamental if the corresponding strain fields overlap. This interaction strength is given by the specific sine-dependence in Eq. (28).

$$\varepsilon_{3p} = C_3 \varepsilon_p^3,$$  \hspace{1cm} (32)

where $C_3$ is the fifth harmonic proportionality factor. Again one notices the very good agreement between the position dependence of the defect on the proportionality factor obtained from the numerical and the analytical results. The sine-dependency of Eq. (29) describes quite well the sensitivity to the position of the nonlinearity.

4. Conclusions

A Nonlinear RUS model was presented for the case of hysteretic nonlinearity. The model predicts the shift in resonance frequency with amplitude, and the dependence of the harmonic amplitudes on the fundamental amplitude. Using these analytical expressions, the influence of the position of the nonlinearity (which is modeling the defect) is studied and compared with numerical results. The availability of closed analytical solutions can be of high importance in inverse modeling, which will turn out to be extremely useful for damage imaging applications.

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6. References